ON THE BEHAVIOR OF p-ADIC EULER ℓ-FUNCTIONS

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ABSTRACT. In this paper we propose a construction of p-adic Euler ℓ-function using Kubota-Leopoldt's approach and Washington's one. We also compute the derivative of p-adic Euler ℓ-function at $s = 0$ and the values of p-adic Euler ℓ-function at positives integers.

1. INTRODUCTION

Let p be an odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-1}$. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the ring of (rational) integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively.

The definition of Euler polynomials is well known and appear in many classical results (cf. [3, 11, 13, 15, 22, 25]). We consider the following generating function

$$(1.1) \qquad F(t, x) = \frac{2e^{xt}}{e^t + 1}.$$

By (1.1) it is easy to get $F(t, x + 1) + F(t, x) = 2e^{xt}, F(t, -x) + F(-t, x) = 2e^{-xt}$, and $F(t, 1 - x) - F(-t, x) = 0$. Expand $F(t, x)$ into a power series of t :

$$(1.2) \qquad F(t, x) = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

The coefficients $E_n(x), n \geq 0$, are called Euler polynomials. The Euler numbers E_n can be readily from (1.1) that $E_n(0) = E_n$ (cf. [13]). It is easy to see that Euler polynomials $E_n(x)$ satisfy the identities

$$(1.3) \qquad \begin{aligned} &E_n(x + 1) + E_n(x) = 2x^n; \\ &E_n(-x) + (-1)^n E_n(x) = 2(-1)^n x^n; \\ &E_n(1 - x) = (-1)^n E_n(x), \end{aligned}$$

where $n \geq 0$. Observe that $E_n + (-1)^n E_n = 0, E_n = (-1)^n E_n(1)$, $E_n(-1/2) = (-1)^n 2^{1-n}$ for $n \geq 1$, and $E_n(1/2) = 0$ if n is odd. This yields the values $E_0 = 1, E_1 = -1/2, E_3 = 1/4, E_5 = -1/2, E_7 = 17/8, E_9 = -31/2, E_{11} = 691/4, \ldots$, with $E_{2k} = 0$ for $k = 1, 2, \ldots$.

For a primitive Dirichlet character χ with an odd conductor $f = f_\chi$, the formal power series $F_\chi(t)$ are defined by

$$(1.4) \qquad F_\chi(t) = 2 \sum_{a=1}^{f} \frac{(-1)^a \chi(a) e^{at}}{e^{ft} + 1}, \quad |t| < \frac{\pi}{f}$$

(see e.g. [3, 9, 13, 15, 25, 22, 23]). Generalized Euler numbers $E_{n,\chi}$ belong to the Dirichlet character χ are defined by

$$(1.5) \qquad F_\chi(t) = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}.$$

Let $\mathbb{Q}(\chi)$ denote the field generated over $\mathbb{Q}$ by all the values $\chi(a), a \in \mathbb{Z}$. Then it can be shown that $E_{n,\chi} \in \mathbb{Q}(\chi)$ for each $n \geq 0$. In the complex case, the generating function $F_\chi(t)$ is given by

$$(1.6) \qquad F_\chi(t) = 2 \sum_{a=1}^{f} (-1)^a \chi(a) \sum_{k=0}^{\infty} (-1)^k e^{(a+fk)t} = \sum_{n=0}^{\infty} \left(2 \sum_{l=1}^{\infty} (-1)^l \chi(l) l^n \right) \frac{t^n}{n!}.$$

Of course, none of sums above are convergent and the argument is not rigorous. However, the argument can be made rigorous in the analytic continuation when the series (1.4) and (1.6) have a common domain of convergence complex plane (cf. [2]). Comparing coefficients of $t^n/n!$ on both sides of (1.5) and (1.6) gives

$$(1.7) \qquad E_{n,\chi} = 2 \sum_{l=1}^{\infty} (-1)^l \chi(l) l^n$$

(see [13, Theorem 7]).

This is used to construct the Euler ℓ-functions attached to χ.

Let χ be a primitive Dirichlet character with an odd conductor f. Define the Euler ℓ-function attached to χ by

$$(1.8) \qquad \ell_E(s, \chi) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n^s},$$

where $\text{Re}(s) > 0$ (cf. [13, 14, 15]). The Euler ℓ-function attached to χ can be continued into the complex plane. It is believed that the analysis of Euler ℓ-function $\ell_E(s, \chi)$ attached to χ began with Euler's study of the zeta function $\zeta(s)$ in which he considered the function only for real values of s (cf. [1]). In particular, if we substitute $\chi = \chi^0$, the trivial character, in (1.8), we have put

$$(1.9) \qquad \zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

(cf. [1, 13, 14]). The term $1 - 2^{1-s}$ is zero only at the isolated points $s_n = 1 + 2\pi i n / \log(2)$ for n any integer. So, we can use $\zeta_E(s)$ to define $\zeta(s)$ on this larger set by defining

$$(1.10) \qquad \zeta_E(s) = -2(1 - 2^{1-s})\zeta(s) \quad \text{for } \text{Re}(s) > 0, \ s \neq s_n.$$

Here, $\zeta(s)$ denotes the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (see [1, 16, 19, 24]). The function $\zeta_E(s)$ is called the Euler zeta function (cf. [13, 15]).

Recall that the Dirichlet L-function attached to χ is defined by

$$(1.11) \qquad L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This function can be continued analytically to the entire complex plane, except for a simple $s = 1$ when $\chi = \chi^0$, in which case we have the Riemann zeta function, $\zeta(s) = L(s, \chi^0)$. It is known that the values of $L(s, \chi)$ for $s \in \mathbb{Z}$ with $s \leq 0$ are algebraic numbers so that they may be considered as elements of the field $\overline{\mathbb{Q}}_p$ (see [9]).

The existence of the p-adic analogue of Dirichlet L-function was, in fact, proved by Kubota-Leopoldt [17], and they called it the p-adic L-function for the character χ. Nowadays, p-adic L-functions are quite important in number theory in particular by the works of Iwasawa [9], and different properties of these functions have been investigated by various authors, and are strictly related, by methods and applications, to many works appeared in the recent literature, where different analytic properties of various type of p-adic L-functions are investigated (see [5, 6, 7, 9, 12, 13, 16, 17, 20, 21, 23]).

Recently, p-adic interpolation functions of Euler numbers have been treated by Tsumura [22], Kim [13], Kim et al. [15], Young [25]. In [13], Kim showed several properties and derived formulas involving the generalized Euler numbers with a parameter. Simsek [21] studied twisted (h, q)-Bernoulli numbers and polynomials, and (h, q)-Euler numbers and polynomials.

The purpose of this paper is to give a construction of p-adic Euler ℓ-function which interpolates the values $(1 - \chi_n(p)p^n)\ell_E(-n, \chi_n)$ for $n \geq 0$ using Kubota-Leopoldt's method [17] (see Section 3 for definitions). Let us recall that another construction was done in [15]. We also compute the derivative of p-adic Euler ℓ-function at $s = 0$ and the values of p-adic Euler ℓ-function at positives integers.

2. Preliminaries

Generalized Euler numbers play fundamental roles in various branches of mathematics including combinatorics, number theory, special functions and analysis (cf. [1, 2, 9, 16, 24]). Now, we present some of the fundamental properties of generalized Euler numbers which are need in the later sections.

By (1.1), (1.2), (1.4) and (1.5), the generalized Euler numbers satisfy the relation

$$(2.1) \qquad E_{k,\chi} = f^k \sum_{a=1}^{f} (-1)^a \chi(a) E_k \left(\frac{a}{f} \right).$$

In particular, $E_{0,\chi} = \sum_{a=1}^{f} (-1)^a \chi(a)$ for all χ. From (1.6) and (1.8) we can deduce the formula

$$(2.2) \qquad 2\sum_{n=1}^{\infty} (-1)^n \chi(n) e^{-nt} = \sum_{k=0}^{\infty} \ell_E(-k, \chi) \frac{(-t)^k}{k!}$$

by considering the vertical line integral $\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} t^{-s} \Gamma(s) \ell_E(s, \chi) ds$ and moving the path integration to $\mathrm{Re}(s) = -\infty$ (see [19, Eq. (3.9)]). The left hand side of

(2.2) can be simplified as follows:

$$2\sum_{n=1}^{\infty}(-1)^n\chi(n)e^{-nt} = 2\sum_{a=1}^{f}(-1)^a\chi(a)e^{-at}\sum_{k=0}^{\infty}(-1)^k e^{-fkt}$$

(2.3)

$$= \sum_{k=0}^{\infty}\left(f^k\sum_{a=1}^{f}(-1)^a\chi(a)E_k\left(1-\frac{a}{f}\right)\right)\frac{t^k}{k!}.$$

Using the identity $E_k(1-a/f) = (-1)^k E_k(a/f)$ and (2.3), we observe that

$$(2.4) \qquad\qquad \ell_E(-k,\chi) = f^k\sum_{a=1}^{f}(-1)^a\chi(a)E_k\left(\frac{a}{f}\right).$$

From (2.1) we can rewrite (2.4) as

$$(2.5) \qquad\qquad\qquad\qquad \ell_E(-n,\chi) = E_{n,\chi}$$

for $n \geq 0$ (cf. [13, 15]). If $\chi = \chi^0$, we have

$$(2.6) \qquad\qquad\qquad\qquad \zeta_E(-n) = E_n$$

when $n \geq 1$. Further, if $n = 0$, $\zeta_E(0) = -2(1 - 1 + 1 - 1 + \cdots) = -E_0$ (see [1, p. 189, (2)]). It is clear from (1.4) that $F_\chi(-t) = -\chi(-1)F_\chi(t)$, if $\chi \neq \chi^0$, the trivial character. Hence

$$(2.7) \qquad\qquad (-1)^{n+1}E_{n,\chi} = \chi(-1)E_{n,\chi}, \quad n \geq 0.$$

In particular we obtain

$$(2.8) \qquad\qquad E_{n,\chi} = 0 \quad \text{if } \chi \neq \chi^0,\ n \not\equiv \delta_\chi \pmod 2,$$

where $\delta_\chi = 0$ if $\chi(-1) = -1$ and $\delta_\chi = 1$ if $\chi(-1) = 1$. However, since $F_{\chi^0}(t) = -2e^t/(e^t+1)$, we easily see that $E_{n,\chi^0} = E_n$ if n odd and 0 if n even, and $E_{0,\chi^0} = -E_0 = -1$. From (2.5) and (2.7), we may deduce that

$$(2.9) \qquad\qquad \ell_E(-n,\chi) = (-1)^{n+1}\chi(-1)\ell_E(-n,\chi), \quad n \geq 0.$$

For $n \geq 0$ therefore, $\ell_E(-n,\chi) \neq 0$ if and only if $\chi(-1) = (-1)^{n+1}$. That is, $\ell_E(-n,\chi) \neq 0$ if and only if χ and $n+1$ have the same parity.

Let $\mathbb{Q}_p(\chi)$ denote the field generated over $\mathbb{Q}_p$ by $\chi(a), a \in \mathbb{Z}$ (in an algebraic closure of $\mathbb{Q}_p$). $\mathbb{Q}_p(\chi)$ is a locally compact topological field containing $\mathbb{Q}(\chi)$ as a dense subfield. We can state the following lemma.

Lemma 2.1. *If $n \in \mathbb{Z}, n \geq 0$, then there exist a Witt's formula of $E_{n,\chi}$ in $\mathbb{Q}_p(\chi)$ such that*

$$E_{n,\chi} = \lim_{N\to\infty}\sum_{a=1}^{fp^N}(-1)^a\chi(a)a^n.$$

Herein as usual we set $\chi(a) = 0$ if a is not prime to the conductor f.

Proof. For $t \in \overline{\mathbb{Q}}_p$ with $|t|_p < |p|_p^{\frac{1}{p-1}}$, we have

$$\lim_{N\to\infty}\sum_{b=0}^{p^N-1}(-1)^b e^{bt} = \lim_{N\to\infty}\frac{1+e^{p^N t}}{1+e^t}$$

(see [16]). As $\lim_{N\to\infty} e^{p^N t} = 1$, we obtain

$$(2.10) \qquad \lim_{N\to\infty} \sum_{b=0}^{p^N-1} (-1)^b e^{bt} = \frac{2}{1+e^t},$$

so

$$(2.11) \qquad \lim_{N\to\infty} \sum_{a=1}^{fp^N} (-1)^a \chi(a) e^{ta} = \lim_{N\to\infty} \sum_{a=1}^{f} (-1)^a \chi(a) \sum_{b=0}^{p^N-1} (-1)^b (e^{tf})^b e^{at}$$

$$= F_\chi(t),$$

where χ is a primitive Dirichlet character with an odd conductor $f = f_\chi$. Using (1.4) and comparing the coefficients of $t^n/n!$ on both sides of (2.11), we obtain the result. $\qquad\square$

Corollary 2.2. *In* $\mathbb{Q}_p(\chi)$,

$$(1 - \chi(p)p^n)E_{n,\chi} = \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a) a^n.$$

We define the difference operator Δ_c by the forward difference operator

$$(2.12) \qquad \Delta_c a_n = a_{n+c} - a_n.$$

Repeated application of this operator can be expressed in the form

$$(2.13) \qquad \Delta_c^k a_n = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a_{n+jc}$$

for some positive $k \in \mathbb{Z}$ (see [25]). An application of (2.13) to the sequence $\{E_{n,\chi}\}$ yields

$$(2.14) \qquad \Delta_{p-1}^k E_{n,\chi} = \lim_{N\to\infty} \sum_{a=1}^{fp^N} (-1)^a \chi(a) a^n (a^{p-1} - 1)^k,$$

where $n, k \geq 1$. Therefore, we obtain the congruence in the field $\mathbb{Q}_p(\chi)$:

$$(2.15) \qquad \Delta_{p-1}^k E_{n,\chi} \equiv 0 \pmod{p^k}$$

for all natural numbers n such that $n \geq k$. This is an analogue to Kummer's congruences for the ordinary Bernoulli numbers B_n. Therefore we obtain the following theorem.

Theorem 2.3. *If n be the natural numbers such that $n \geq k$. Then*

$$\Delta_{p-1}^k E_{n,\chi} \equiv 0 \pmod{p^k}.$$

Remark 2.4. Shiratani [20, Eq. (10)] has noted that

$$\Delta_{p-1}^k \frac{1}{n} B_n \equiv 0 \pmod{p^k}$$

for all natural numbers n such that $n \geq k + 1$. This congruence is the most well-known formula of Kummer (cf. [8, 25]).

3. CONSTRUCTION OF p-ADIC EULER ℓ-FUNCTIONS

In this section, we investigate several interesting properties of p-adic Euler ℓ-functions interpolates the generalized Euler numbers in methods similar to [9, §3] and [24, §5.3].

For each $a \in \mathbb{Z}_p$ with $p \nmid a$, a can be uniquely written in the form

$$(3.1) \qquad\qquad \langle a \rangle = \omega^{-1}(a)a,$$

where ω is the Teichmüller character. Then we have $\langle a \rangle \equiv 1 \pmod{p\mathbb{Z}_p}$. For $p > 2$, $\lim_{n\to\infty} a^{p^n} = \omega(a)$ (see [24]).

Let χ be the Dirichlet character with an odd conductor $f = f_\chi$. For $n \geq 1$, we define χ_n to be the primitive character associated with the character

$$\chi_n : (\mathbb{Z}/\text{l.c.m.}(f,p)\mathbb{Z})^\times \to \mathbb{C}^\times$$

defined by $\chi_n(a) = \chi(a)\omega^{-n}(a)$. Let $\mathbb{Q}_p(\chi)$ be the field generated over $\mathbb{Q}_p$ by the values $\chi(a), a \in \mathbb{Z}$. We define a sequence of elements $\epsilon_{n,\chi}, n \geq 0$, in $\mathbb{Q}_p(\chi)$ by

$$(3.2) \qquad\qquad \epsilon_{n,\chi} = (1 - \chi_n(p)p^n)E_{n,\chi_n},$$

where E_{n,χ_n} is the generalized Euler number defined in (1.5). Note that $\chi_n(a)$ is in $\mathbb{Q}_p(\chi)$ for any $n \geq 0$ and $a \in \mathbb{Z}$.

We put

$$\mathcal{D} = \left\{ s \in \mathbb{C}_p \ : \ |s|_p < |p|_p^{-(p-2)/(p-1)} \right\}.$$

Now we define an interpolation function $\ell_{p,E}(s,\chi)$ for generalized Euler numbers over $\mathcal{D}$ by

$$(3.3) \qquad\qquad \ell_{p,E}(s,\chi) = \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a)\langle a \rangle^{1-s}$$

(cf. [9, 12, 23, 24]). The p-adic function $\ell_{p,E}(s,\chi)$ will be called the p-adic Euler ℓ-function. From Lemma 2.1 and (3.1) we easily see that

$$(3.4)$$
$$\ell_{p,E}(1-n,\chi) = \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a)\langle a \rangle^n$$
$$= \lim_{N\to\infty} \left(\sum_{a=1}^{fp^N} (-1)^a \chi(a)\langle a \rangle^n - \sum_{a=1}^{fp^{N-1}} (-1)^{ap} \chi(ap)\langle ap \rangle^n \right)$$
$$= \epsilon_{n,\chi}, \quad n \geq 1.$$

Theorem 3.1. *Let n, c, k be positive integers with $c \equiv 0 \pmod{p-1}, p > 2$. Then*

$$\Delta_c^k \epsilon_{n,\chi} \equiv 0 \pmod{p^k \mathbb{Z}_p[\chi]}.$$

Proof. From (2.13) and (3.4), we note that

$$\Delta_c^k \epsilon_{n,\chi} = \Delta_c^k \ell_{p,E}(1-n,\chi)$$

$$= \sum_{j=0}^k \binom{k}{j}(-1)^{k-j} \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a)\langle a\rangle^{n+jc}$$

$$= \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a)(\langle a\rangle^c - 1)^k \langle a\rangle^n.$$

If $c \equiv 0 \pmod{p-1}$, then since $\langle a\rangle \in \mathbb{Z}_p^\times$, we have

$$(\langle a\rangle^c - 1)^k \equiv 0 \pmod{p^k \mathbb{Z}_p}$$

by Euler's Theorem. We thus conclude that $\Delta_c^k \epsilon_{n,\chi} \equiv 0 \pmod{p^k \mathbb{Z}_p[\chi]}$. $\square$

Corollary 3.2. *Let $n \equiv n' \pmod{p-1}$ and $c \equiv 0 \pmod{p-1}$. Then*

$$\Delta_c^k \epsilon_{n,\chi} \equiv \Delta_c^k \epsilon_{n',\chi} \pmod{p^{k+1}\mathbb{Z}_p[\chi]}.$$

Proof. Note that

$$\Delta_c^k \epsilon_{n,\chi} - \Delta_c^k \epsilon_{n',\chi} = \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a)(\langle a\rangle^c - 1)^k (\langle a\rangle^{n-n'} - 1)\langle a\rangle^{n'}.$$

If $n - n' \equiv 0 \pmod{p-1}$, then we have $\langle a\rangle^{n-n'} \equiv 1 \pmod{p\mathbb{Z}_p}$. Thus the result follows from Theorem 3.1. $\square$

Lemma 3.3 ([9, p. 19, Lemma 1]). *If $A(x), B(x) \in \mathbb{Q}_p(\chi)[[x]]$, convergent in a neighborhood of 0 in $\overline{\mathbb{Q}}_p$ and $A(\xi_n) = B(\xi_n)$ for a sequence of elements $\xi_n \neq 0, n \geq 0$, in $\overline{\mathbb{Q}}_p$ such that $\lim_{n\to 0} \xi_n = 0$, then $A(x) = B(x)$.*

Let

(3.5)
$$c_n = \sum_{i=0}^n \binom{n}{i}(-1)^{n-i}\epsilon_{i,\chi}, \quad n \geq 0.$$

Lemma 3.4 (cf. [9, p. 26, Lemma 4]).

$$|c_n|_p \leq |p^n|_p, \quad n \geq 0.$$

Proof. From the definition of $\epsilon_{i,\chi}$ and (3.4), we get

$$c_n = \sum_{i=0}^n \binom{n}{i}(-1)^{n-i} \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a)\langle a\rangle^i$$

$$= \lim_{N\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a)(\langle a\rangle - 1)^n.$$

Since $\langle a \rangle \equiv 1 \pmod{p}$ then $(\langle a \rangle - 1)^n \equiv 0 \pmod{p^n}$, hence we can conclude

$$|c_n|_p = \lim_{N \to \infty} \left| \sum_{\substack{a=1 \\ p \nmid a}}^{f p^N} (-1)^a \chi(a)(\langle a \rangle - 1)^n \right|_p$$

$$= \lim_{N \to \infty} |p^n \theta_n(N)|_p \le |p^n|_p,$$

because

$$\sum_{\substack{a=1 \\ p \nmid a}}^{f p^N} (-1)^a \chi(a)(\langle a \rangle - 1)^n = p^n \theta_n(N)$$

for some $\theta_n(N)$ with $|\theta_n(N)|_p \le 1$. This is the desired conclusion. $\qquad \square$

We now apply Theorem 1 in [9, p. 22] for the above sequences $\epsilon_{n,\chi}$ and c_n, $n \ge 0$, in $\mathbb{Q}_p(\chi)$ and for

$$r = |p|_p < |p|_p^{1/(p-1)}.$$

Thus we show that there exists such $A_\chi(x) \in \mathbb{Q}_p(\chi)[[x]]$ convergent for $|\xi|_p < |p|_p^{1/(p-1)}|p|_p^{-1} = |p|_p^{-(p-2)/(p-1)}$ which takes the prescribed values at the non-negative integers,

(3.6) $$A_\chi(n) = \epsilon_{n,\chi}.$$

Let

(3.7) $$\ell_{p,E}(s, \chi) = A_\chi(1 - s)$$

with the $A_\chi(x)$ mentioned above. The uniqueness of $\ell_{p,E}(s, \chi)$ is a consequence of Lemma 3.3. Then we have the following theorem.

Theorem 3.5. *Let χ be a Dirichlet character with an odd conductor $f = f_\chi$. Then there exists a p-adic analytic function*

$$\ell_{p,E}(s, \chi) = \sum_{n=0}^{\infty} (-1)^n a_n (s - 1)^n, \quad a_n \in \mathbb{Q}_p(\chi)$$

defined on $\mathcal{D}$ such that

$$\ell_{p,E}(1 - n, \chi) = \epsilon_{n,\chi}, \quad n \ge 1.$$

Moreover, the p-adic Euler ℓ-function $\ell_{p,E}(s, \chi)$ interpolates the numbers $(1 - \chi_n(p)p^n)\ell_E(-n, \chi_n)$ for $n \ge 0$.

Remark 3.6. Since $\chi_n = \chi$ whenever $n \equiv 0 \pmod{p-1}$, we have

$$\ell_{p,E}(1 - n, \chi) = 0 \quad \text{if } \chi \ne \chi^0, \ n \not\equiv \delta_\chi \pmod 2, \ n \equiv 0 \pmod{p-1}.$$

Corollary 3.7. *Suppose $\chi \ne 1$ and $p^2 \nmid f_\chi$. Then*

$$\ell_{p,E}(s, \chi) = a_0 - a_1(s - 1) + a_2(s - 1)^2 - \cdots$$

with $|a_0|_p \le 1$ and with $p \mid a_n$ for all $n \ge 1$.

Proof. By using (3.3) and the formula

$$\langle a \rangle^{1-s} = \exp((1 - s)\log_p \langle a \rangle) = \sum_{n=0}^{\infty} \frac{1}{n!}(1 - s)^n (\log_p \langle a \rangle)^n,$$

where exp (resp. $\log_p$) is the p-adic exponential (resp. logarithm) function (see [9]), we have

$$\ell_{p,E}(s,\chi) = \sum_{n=0}^{\infty}(-1)^n(s-1)^n \lim_{N\to\infty}\sum_{\substack{a=1 \\ p\nmid a}}^{fp^N}(-1)^a\chi(a)\frac{(\log_p\langle a\rangle)^n}{n!}, \quad n \geq 0.$$

From Theorem 3.5, we find that

$$a_n = \lim_{N\to\infty}\sum_{\substack{a=1 \\ p\nmid a}}^{fp^N}(-1)^a\chi(a)\frac{(\log_p\langle a\rangle)^n}{n!}, \quad n \geq 0.$$

If $n \geq 1$, then $(\log_p\langle a\rangle)^n/n! \equiv 0 \pmod{p}$. Thus we obtain

$$p \mid a_n \quad \text{for } n \geq 1.$$

By Lemma 3.4 $|a_0|_p = |c_0|_p \leq 1$ is obvious. $\qquad\square$

By Corollary 3.7, we have

$$(3.8) \qquad \ell_{p,E}(s,\omega^m) = \ell_{p,E}(s,\omega^n)$$

if $m \equiv n \not\equiv 0 \pmod{p-1}$, hence we have the following which can be proved in the same way as Corollary 5.14 in [24].

Corollary 3.8. *If m and n are positive even integers with $m \equiv n \pmod{(p-1)p^k}$ and $n \not\equiv 0 \pmod{p}$, then*

$$(1-p^m)E_m \equiv (1-p^n)E_n \pmod{p^{k+1}}.$$

Let χ be the Dirichlet character with an odd conductor $f = f_\chi \in \mathbb{N}$. Let F be a positive integer multiple of p odd and f. Then by (1.6), we have

$$(3.9) \qquad F_\chi(t) = 2\sum_{m=0}^{\infty}(-1)^m\chi(m)e^{mt} = 2\sum_{a=1}^{F}(-1)^a\chi(a)\frac{(e^{\frac{a}{F}})^{Ft}}{e^{Ft}+1}.$$

Therefore, by (1.1), (1.2) and (1.5), we obtain the following

$$(3.10) \qquad E_{n,\chi} = F^n\sum_{a=1}^{F}(-1)^a\chi(a)E_n\left(\frac{a}{F}\right).$$

If $\chi_n(p) \neq 0$, then $(p,f_{\chi_n}) = 1$, so that F/p is a multiple of f_{χ_n}. From (3.10), we derive

$$\chi_n(p)p^n E_{n,\chi_n} = \chi_n(p)p^n\left(\frac{F}{p}\right)^n\sum_{a=1}^{F/p}(-1)^a\chi_n(a)E_n\left(\frac{a}{F/p}\right)$$

$$(3.11)$$

$$= F^n\sum_{\substack{a=1 \\ p\mid a}}^{F}(-1)^a\chi_n(a)E_n\left(\frac{a}{F}\right).$$

Thus by (3.2), (3.10) and (3.11), we have

$$(3.12) \qquad \epsilon_{n,\chi} = F^n\sum_{\substack{a=1 \\ p\nmid a}}^{F}(-1)^a\chi_n(a)E_n\left(\frac{a}{F}\right).$$

Since $E_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} E_k$ and $\chi_n(a) = \chi(a)\omega^{-n}(a)$, by (3.12), we easily see that

$$(3.13) \qquad \epsilon_{n,\chi} = \sum_{\substack{a=1 \\ p \nmid a}}^{F} (-1)^a \chi(a) \langle a \rangle^n \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{F}{a}\right)^k E_k,$$

where $\langle a \rangle = \omega^{-1}(a)a$. From Theorem 3.5 and (3.13), we obviously have

$$(3.14) \qquad \ell_{p,E}(-n, \chi) = \sum_{\substack{a=1 \\ p \nmid a}}^{F} (-1)^a \chi(a) \langle a \rangle^{1+n} \sum_{k=0}^{\infty} \binom{1+n}{k} \left(\frac{F}{a}\right)^k E_k$$

for $n \geq 0$ (cf. [13, 15, 10, 23]). Using Theorem 3.5 and (3.14), our p-adic Euler ℓ-function $\ell_{p,E}(s, \chi)$ can be written in the form

Theorem 3.9. *Let χ be a Dirichlet character with an odd conductor $f = f_\chi$, and let F be a positive integer multiple of p odd and f. Then*

$$\ell_{p,E}(s, \chi) = \sum_{\substack{a=1 \\ p \nmid a}}^{F} (-1)^a \chi(a) H_p(s, a, F), \quad s \in \mathbb{Z}_p.$$

Here $H_p(s, a, F)$ is the Washington function [24, §5.2]. It is defined for $a \in \mathbb{Z}_p^{\times}$, $s \in \mathbb{Z}_p$ and $p \mid F$ by

$$H_p(s, a, F) = \langle a \rangle^{1-s} \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{a}\right)^k E_k.$$

This function is analytic function for $s \in \mathbb{Z}_p$.

Proof. The left hand side and right the hand side have the same values at the negative integers, which are dense in the ring of p-adic integers $\mathbb{Z}_p$, and they are both analytic, hence they coincide. $\qquad\square$

Next, we derive some functional equations for the Washington function (cf. [10]).

Proposition 3.10. *If $s \in \mathbb{Z}_p$ and $a \in \mathbb{Z}_p^{\times}$, then*

$$H_p(s, a, F) = H_p(s, F - a, F).$$

Proof. From the definition of $H_p(s, a, F)$, we obtain

$$H_p(1-n, a, F) = \langle a \rangle^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{F}\right)^{-k} E_k = \langle a \rangle^n E_n \left(\frac{a}{F}\right),$$

where $n \geq 1, a \in \mathbb{Z}_p^{\times}$ and $p \mid F$ (cf. [23]). One can see easily from this that

$$H_p(1-n, F-a, F) = \langle F-a \rangle^n \left(\frac{F}{F-a}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F-a}{F}\right)^{n-k} E_k$$

$$= \langle F-a \rangle^n \left(\frac{F}{F-a}\right)^n E_n \left(1 - \frac{a}{F}\right)$$

$$= \langle a \rangle^n \langle -1 \rangle^n \left(\frac{F}{a}\right)^n E_n \left(\frac{a}{F}\right)$$

$$= H_p(1-n, a, F),$$

since $E_n(1-x) = (-1)^n E_n(x)$ and $\omega^{-1}(F-a) = \omega^{-1}(-a) = -\omega^{-1}(a)$ (cf. [9]). Using the fact that the set of non-negative integers are dense in $\mathbb{Z}_p$, we see that

$$H_p(s, a, F) = H_p(s, F-a, F)$$

for $s \in \mathbb{Z}_p$ and $a \in \mathbb{Z}_p^{\times}$. This is the desired conclusion. $\qquad\square$

4. EVALUATION OF $\ell'_{p,E}(0, \chi)$ AND $\ell_{p,E}(n, \chi)$ AT $n \geq 1$

In finding the value of the derivation of the function $\ell_{p,E}(s, \chi)$ at $s = 0$ and the values of $\ell_{p,E}(n, \chi)$ at $n \geq 1$ (cf. [5, 6]), we define the function

$$(4.1) \qquad G_{p,E}(x) = \lim_{N \to \infty} \sum_{a=0}^{p^N - 1} \left\{ (x+a) \log_p(x+a) - (x+a) \right\} (-1)^a$$

for $x \in \mathbb{C}_p$ with $|x|_p > 1$ (cf. [13]). This definition is slightly different from the original one due to Diamond (cf. [4, p. 326, Definition of G_p]). Here $\log_p$ is the p-adic logarithm function of Iwasawa (see [9]). Let $|x|_p > 1$. For $a \in \mathbb{Z}_p$ we have $\left| \frac{a}{x} \right|_p < 1$ so that

$$(x+a) \log_p(x+a) - (x+a)$$
$$= x \left(1 + \frac{a}{x} \right) \log_p \left(1 + \frac{a}{x} \right) + (x+a) \log_p(x) - (x+a)$$
$$= a + x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \left(\frac{a}{x} \right)^{n+1} + (x+a) \log_p(x) - (x+a)$$

(cf. [4]). From (2.10), we easily obtain that

$$(4.2) \qquad \lim_{N \to \infty} \sum_{a=0}^{p^N - 1} a^n (-1)^a = E_n,$$

so

$$(4.3) \qquad \begin{aligned} G_{p,E}(x) &= \left(x - \frac{1}{2} \right) \log_p(x) - x - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \frac{1}{x^n} E_{n+1} \\ &= \left(x - \frac{1}{2} \right) \log_p(x) - x - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \frac{1}{x^n} \zeta_E(-n-1), \end{aligned}$$

where we use (2.6) and the fact that $E_{n+1} = 0$ if n is odd. This formula arises from the asymptotic expansion of the classical complex log gamma function. We put

$$D = \frac{\mathrm{d}}{\mathrm{d}x}.$$

Furthermore, if D^m denotes the m-th derivative, by (4.3), we have

$$(4.4) \qquad D^m G_{p,E}(x) = (-1)^m (m-2)! \sum_{k=0}^{\infty} \binom{-m+1}{k} \frac{1}{x^{m+k-1}} E_k, \quad m \geq 2.$$

Proposition 4.1. *Let p be a fixed odd prime number. Then*
  (1) $G_{p,E}(1-x) + G_{p,E}(x) = 0$ $(x \in \mathbb{C}_p - \mathbb{Z}_p)$.
  (2) $G_{p,E}(x) - G_{p,E}(-x) = 2x(\log_p(x) - 1)$ $(x \in \mathbb{C}_p - \mathbb{Z}_p)$.
  (3) $G_{p,E}(1+x) + G_{p,E}(x) = 2x(\log_p(x) - 1)$ $(x \in \mathbb{C}_p - \mathbb{Z}_p)$.

Proof. Upon expanding either side of Proposition 3.10 in powers of s and equating the coefficients of s, we get

$$\omega^{-1}(a)\left(\frac{F}{2}-a\right)(1+\log_p(F))-\omega^{-1}(a)FG_{p,E}\left(\frac{a}{F}\right)$$

$$=\omega^{-1}(F-a)\left(\frac{F}{2}-(F-a)\right)(1+\log_p(F))-\omega^{-1}(F-a)FG_{p,E}\left(\frac{F-a}{F}\right).$$

This yields

$$G_{p,E}\left(\frac{a}{F}\right)+G_{p,E}\left(1-\frac{a}{F}\right)=0.$$

This is true for any positive integer F divisible by p, and any p-unit a, thus established (1). Using $\log_p(-x)=\log_p(x)$, (2) is straightforward consequences of the the power series expansion. Formula (3) follows form (1) and (2). $\qquad\square$

Now we prove a formula for $\ell'_{p,E}(0,\chi)$ which is analogous to a classical formula of $L'_p(0,\chi)$ (see [4, 6]).

Theorem 4.2. *Let χ be a primitive Dirichlet character, and let F be a positive odd integral multiple of p and f_χ. Then*

$$\ell'_{p,E}(0,\chi)=F\sum_{\substack{a=1\\p\nmid a}}^{F}(-1)^{a+1}\chi_1(a)G_{p,E}\left(\frac{a}{F}\right)-(1+\log_p(F))\ell_{p,E}(0,\chi).$$

Proof. We have the expansions (cf. [7]):

$$\langle a\rangle^{1-s}=\langle a\rangle(1-s\log_p\langle a\rangle+\cdots$$

$$\binom{1-s}{n}=\frac{(-1)^{n+1}}{n(n-1)}s+\cdots,$$

provided $n\geq 2$. From these expansions and Theorem 3.9, we find that the coefficient of s in $\ell_{p,E}(s,\chi)$ is

$$\sum_{\substack{a=1\\p\nmid a}}^{F}(-1)^a\chi_1(a)\left(\frac{F}{2}-\left(a-\frac{F}{2}\right)\log_p\langle a\rangle+F\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{F}{a}\right)^n E_{n+1}\right).$$

Furthermore, from (4.3), we have

$$G_{p,E}\left(\frac{a}{F}\right)=\left(\frac{a}{F}-\frac{1}{2}\right)\log_p\left(\frac{a}{F}\right)-\frac{a}{F}-\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{F}{a}\right)^n E_{n+1}.$$

Since the value of $\ell'_{p,E}(0,\chi)$ is the coefficient of s in the expansion of $\ell_{p,E}(s,\chi)$ about $s=0$, by evaluating the sum

$$\sum_{\substack{a=1\\p\nmid a}}^{F}(-1)^a\chi_1(a)\left(a-\frac{F}{2}\right)=\sum_{a=1}^{F}(-1)^a\chi_1(a)\left(a-\frac{F}{2}\right)$$

$$-\chi_1(p)p\sum_{a=1}^{F/p}(-1)^a\chi_1(a)\left(a-\frac{\frac{F}{p}}{2}\right)$$

$$=(1-\chi_1(p)p)E_{1,\chi_1}$$

$$=\ell_{p,E}(0,\chi),$$

we obtain the result. $\qquad\square$

We can now obtain the values of p-adic Euler ℓ-functions at positives integers (cf. [5]).

Theorem 4.3. *Let χ be a primitive Dirichlet character with an odd conductor $f = f_\chi$. Then*

$$\ell_{p,E}(n, \chi_{n-1}) = \frac{(-pf)^{-n+1}}{(n-2)!} \sum_{\substack{a=1 \\ p\nmid a}}^{fp} (-1)^a \chi(a) \left(D^n G_{p,E}\right)\left(\frac{a}{pf}\right), \quad n \geq 2.$$

In particular, $\ell_{p,E}(1, \chi) = (1 - \chi(p))E_{0,\chi}$.

Proof. Note that

$$(4.5) \qquad \frac{1}{(x+a)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^{-n-k} a^k.$$

Applying (4.2), (4.4) and (4.5), we have

$$(4.6) \qquad D^n G_{p,E}(x) = (-1)^n (n-2)! \lim_{N \to \infty} \sum_{a=0}^{p^N - 1} (-1)^a (x+a)^{-n+1}, \quad n \geq 2.$$

Furthermore, if $n \geq 2$ and χ is a primitive Dirichlet character with an odd conductor $f = f_\chi$, then by (3.3), we obtain

$$\ell_{p,E}(n, \chi_{n-1}) = \lim_{N \to \infty} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^N} (-1)^a \chi(a) \langle a \rangle^{-n+1}$$

$$= \sum_{\substack{a=1 \\ p\nmid a}}^{fp} (-1)^a \chi(a) \lim_{N \to \infty} \sum_{b=0}^{p^{N-1}-1} (-1)^b (a + pfb)^{-n+1}.$$

We combine this formula with (4.6) to obtain our result. $\qquad\square$

5. Further remarks and observations

The Euler measure is defined for each positive integer n by (see [14, Section 4])

$$(5.1) \qquad \mu_{n,E}(a + p^N \mathbb{Z}_p) = (-1)^a p^{nN} E_n\left(\frac{a}{p^N}\right),$$

where $E_n(x)$ is the Euler polynomials. Note that $\mu_{n,E}(\mathbb{Z}_p) = E_n$ and $\mu_{n,E}(p\mathbb{Z}_p) = p^n E_n$. We deduce that $\mu_{n,E}(\mathbb{Z}_p^\times) = (1 - p^n)E_n$. Let $UD(\mathbb{Z}_p)$ be the set of uniformly differential function on $\mathbb{Z}_p$. When $f \in UD(\mathbb{Z}_p)$, (5.1) allows us to define

$$(5.2) \qquad \int_{\mathbb{Z}_p} f(x) \mu_{n,E}(x) = \lim_{N \to \infty} \sum_{a=0}^{p^N - 1} f(a) \mu_{n,E}(a + p^N \mathbb{Z}_p).$$

Hence, the Euler measures are all related to "fermionic" measure $\mu_{-1}(a + p^N \mathbb{Z}_p) = (-1)^a$ by the property

$$(5.3) \qquad \mu_{n,E}(\mathbb{Z}_p) = \int_{\mathbb{Z}_p} d\mu_{n,E}(x) = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n;$$

$$(5.4) \qquad \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \int_{\mathbb{Z}_p} d\mu_{i,E}(y) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x).$$

(see [11, 13, 14]). This is analogous to the relation between dx^n and dx for n a non-negative integer. Putting these observations together, we have

$$(5.5) \qquad \int_{\mathbb{Z}_p^{\times}} x^n d\mu_{-1}(x) = \mu_{n,E}(\mathbb{Z}_p^{\times}) = (1-p^n)E_n,$$

where $\mathbb{Z}_p^{\times} = \mathbb{Z}_p \setminus p\mathbb{Z}_p$. As we saw in (2.6), the right hand side can be interpreted as

$$(5.6) \qquad\qquad\qquad (1-p^n)\zeta_E(-n)$$

which can be extended to give the p-adic ζ_E-function (see Section 3). Thus, (5.5) should be viewed as expressing the p-adic ζ_E-function as a kind of Mellin transform (cf. [9, 23]). In fact, (5.5) immediately gives the p-adic continuation of the ζ_E-function, as well as the Kummer congruences. Indeed, let $m \equiv n \pmod{(p-1)p^N}$ and $p \nmid x$. From the little Fermat theorem it follows that $x^m \equiv x^n \pmod{p^{N+1}}$. We have

$$|x^m - x^n|_p < \frac{1}{p^{N+1}} \quad \text{for } x \in \mathbb{Z}_p^{\times}.$$

Thus

$$\left| \int_{\mathbb{Z}_p^{\times}} x^m d\mu_{-1}(x) - \int_{\mathbb{Z}_p^{\times}} x^n d\mu_{-1}(x) \right|_p \leq \frac{1}{p^{N+1}}$$

(cf. [12]). We have therefore proved the following theorem (see Corollary 3.8).

Proposition 5.1. *If $(p-1) \nmid n$ and $m \equiv n \pmod{(p-1)p^N}$, then*

$$(1-p^m)E_m \equiv (1-p^n)E_n \pmod{p^{N+1}}.$$

Corollary 5.2. *If s is a non-negative integer and N, n are positive integers respectively such that $(n,p) = 1, N > 0$, then*

$$E_{np^N+s} \equiv E_{np^{N-1}+s} \pmod{p^N}.$$

Proof. For $x \in \mathbb{Z}_p$, we have $x^p \equiv x \pmod{p}$ (Fermat's little theorem). We see by induction that $x^{p^N} \equiv x^{p^{N-1}} \equiv p^N$, and $\sup_{x \in \mathbb{Z}_p} |x^{p^N} - x^{p^{N-1}}|_p \leq p^{-N}$ for $N \geq 1$ (see [18, Lemma 2]). Let $(r,p) = 1$ and let $s \geq 0$. Then we obtain

$$\left| \int_{\mathbb{Z}_p} x^s \left[(x^r)^{p^N} - (x^r)^{p^{N-1}} \right] d\mu_{-1}(x) \right|_p \leq p^{-N},$$

which yields the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

From (5.5), we can see that

$$(5.7) \qquad\qquad |E_n|_p = |1/(1-p^n)|_p \left| \int_{\mathbb{Z}_p^{\times}} x^n d\mu_{-1}(x) \right|_p \leq 1,$$

because the factor before the integral is coprime to p and therefore their p-adic absolute values are 1, and $|\mu_{-1}(U)|_p \leq 1$ for all compact open subsets $U \subset \mathbb{Z}_p^{\times}$. Then we obtain

Proposition 5.3. *If $n \geq 0$, then $|E_n|_p \leq 1$.*

Let d be a fixed positive integer. Let $X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z})$, where the map from $\mathbb{Z}/dp^M\mathbb{Z}$ to $\mathbb{Z}/dp^N\mathbb{Z}$ for $M \geq N$ is a reduction mod dp^N. Let $a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}$ and let

$$(5.8) \qquad X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp^N\mathbb{Z}_p).$$

The generalized Euler numbers $E_{n,\chi}$ can be represented by using fermionic expression of p-adic integral on $\mathbb{Z}_p$ as follows [14]:

$$(5.9) \qquad \begin{aligned} \int_X \chi(x)d\mu_{n,E}(x) &= \int_X \chi(x)x^n d\mu_{-1}(x) = E_{n,\chi}; \\ \int_{pX} \chi(x)d\mu_{n,E}(x) &= \int_{pX} \chi(x)x^n d\mu_{-1}(x) = p^n\chi(p)E_{n,\chi}; \\ \int_{X^*} \chi(x)d\mu_{n,E}(x) &= \int_{X\setminus pX} \chi(x)x^n d\mu_{-1}(x) = (1 - p^n\chi(p))E_{n,\chi}. \end{aligned}$$

Using (5.4) and (5.9), it is not difficult to show that

$$(5.10) \qquad \begin{aligned} E_{n,\chi} &= \int_X \chi(x)x^n d\mu_{-1}(x) \\ &= f^{n-1}\sum_{a=1}^{f}(-1)^a\chi(a)\int_{\mathbb{Z}_p}\left(\frac{a}{f}+x\right)^n d\mu_{-1}(x) \\ &= f^{n-1}\sum_{a=1}^{f}(-1)^a\chi(a)\sum_{i=0}^{n}\binom{n}{i}\left(\frac{a}{f}\right)^{n-i}\int_{\mathbb{Z}_p}x^i d\mu_{-1}(x) \\ &= f^{n-1}\sum_{a=1}^{f}(-1)^a\chi(a)E_n\left(\frac{a}{f}\right), \end{aligned}$$

since $E_n(x) = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}E_i$ (see (2.1)). From (3.3) and (5.9), the p-adic Euler ℓ-function for a Dirichlet character χ can be defined by setting

$$(5.11) \qquad \ell_{p,E}(s,\chi) = \int_{X^*} \chi(x)\langle x\rangle^{1-s}d\mu_{-1}(x) \quad \text{for } s \in \mathbb{Z}_p.$$

From (5.9), we have the following

$$(5.12) \qquad \begin{aligned} \ell_{p,E}(1-n,\chi) &= \int_{X^*}\langle x\rangle^n\chi(x)d\mu_{-1}(x) \\ &= \int_{X^*} x^n\chi_n(x)d\mu_{-1}(x) \\ &= (1 - p^n\chi_n(p))E_{n,\chi_n}. \end{aligned}$$

Using the above expression for $\ell_E(-n,\chi) = E_{n,\chi}, n \geq 0$, we conclude that

$$\ell_{p,E}(1-n,\chi) = (1 - p^n\chi_n(p))\ell_E(-n,\chi_n).$$

This identity can be used to prove the following theorem (see Theorem 3.5).

Theorem 5.4. *There exists a unique p-adic continuous function $\ell_{p,E}(s,\chi), s \in \mathbb{Z}_p$, such that $\ell_{p,E}(1-n,\chi) = (1 - p^n\chi_n(p))\ell_E(-n,\chi_n)$ for $n \geq 1$.*

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